Lecture 16: Excision

Barycentric Subdivision

The fundamental property of homology which makes it computable is excision.

Definition

Let Δ^n be the standard n-simplex with vertices $v_0, \cdots, v_n$. We define its barycenter to be

$$c(\Delta^n) = \frac{1}{n+1} \sum_{i=0}^{n} v_i \in \Delta^n.$$

Definition

We define the barycentric subdivision $\mathscr{B}\Delta^n$ of a n-simplex Δ^n by:

1. $\mathscr{B}\Delta^0 = \Delta^0$.
2. Let $F_0, \cdots, F_n$ be the n-simplexes of faces of Δ^{n+1}, c be the barycenter of Δ^{n+1}. Then $\mathscr{B}\Delta^{n+1}$ consists of $(n+1)$-simplexes with ordered vertices

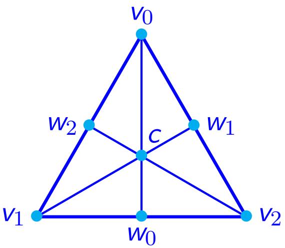
$$[c, w_0, \cdots, w_n]$$

where $[w_0, \cdots, w_n]$ is a n-simplexes in $\mathscr{B}F_0, \cdots, \mathscr{B}F_n$.

Equivalently, a simplex in $\mathscr{B}\Delta^n$ is indexed by a sequence $\{S_0 \subset S_1 \cdots \subset S_n = \Delta^n\}$ where S_i is a face of S_{i+1}. Then its vertices are

$$[c(S_n), c(S_{n-1}), \cdots, c(S_0)].$$

It is seen that Δ^n is the union of simplexes in $\mathscr{B}\Delta^n$.



$$\mathscr{B}\Delta^2 = [c, w_0, v_2] - [c, w_1, v_2] + [c, w_1, v_0] - [c, w_2, v_0] + [c, w_2, v_1] - [c, w_0, v_1]$$

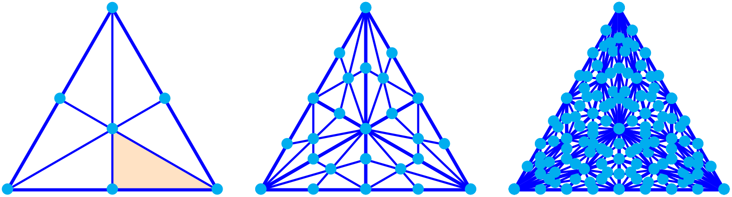


图: Subdivision $\mathscr{B}\Delta^2$, two times $\mathscr{B}^2\Delta^2$, and three times $\mathscr{B}^3\Delta^2$

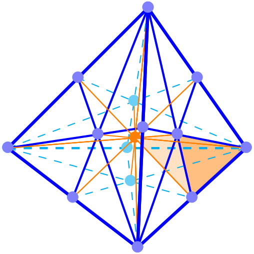


图: Barycentric Subdivision $\mathscr{B}\Delta^3$

Definition

We define the n-chain of barycentric subdivision $\mathscr{B}_n$ by

$$\mathscr{B}_n = \sum_{\alpha} \pm\sigma_\alpha \in S_n(\Delta^n)$$

where the summation is over all sequence

$$\alpha = \{S_0 \subset S_1 \cdots \subset S_n = \Delta^n\}.$$

σ_α is the simplex with ordered vertices

$$[c(S_n), c(S_{n-1}), \cdots, c(S_0)],$$

viewed as a singular n-chain in Δ^n. The sign $\pm$ is about orientation: if the orientation of $[c(S_n), c(S_{n-1}), \cdots, c(S_0)]$ coincides with that of Δ^n, we take $+$; otherwise we take $-$.

Definition

We define the following composition map

$$S_k(\Delta^m) \times S_n(\Delta^k) \to S_n(\Delta^m), \quad \sigma \times \eta \to \sigma \circ \eta.$$

This is defined on generators via the composition

$$\Delta^n \to \Delta^k \to \Delta^m$$

and extended linearly to singular chains.

Similarly, there is a natural map denoted by

$$S_n(\Delta^m) : S_m(X) \to S_n(X), \quad \eta : \sigma \to \eta^*(\sigma) := \sigma \circ \eta$$

where $\eta^*(\sigma) = \sigma \circ \eta$ is the composition of σ with η.

It is easy to see that

$$(\eta_1 \circ \eta_2)^* = \eta_2^* \circ \eta_1^*, \quad \forall \eta_1 \in S_k(\Delta^m), \eta_2 \in S_n(\Delta^k).$$

Example

Let

$$\partial\Delta^n = \sum_{i=0}^{n} (-1)^i \partial^{(i)} \Delta^n \in S_{n-1}(\Delta^n)$$

be the boundary faces. Then

$$(\partial\Delta^n) \circ (\partial\Delta^{n-1}) = 0.$$

The operator

$$\partial_n = (\partial\Delta^n)^* : S_n(X) \to S_{n-1}(x)$$

defines the boundary map in singular chains.

Lemma

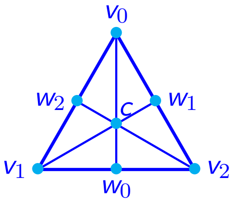
The barycentric subdivision is compatible with the boundary map

$$\partial \mathscr{B}_n = \mathscr{B}(\partial \Delta^n)$$

where $\mathscr{B}(\partial \Delta^n)$ is the barycentric subdivision of faces $\partial \Delta^n$ of Δ^n, viewed as an $(n-1)$-chain in Δ^n. Equivalently, we have the following composition relation

$$\mathscr{B}_n \circ (\partial \Delta^n) = (\partial \Delta^n) \circ \mathscr{B}_{n-1}.$$

The choice of ordering and orientation implies $\partial\mathscr{B}_n = \mathscr{B}(\partial\Delta^n)$. Here is an illustration for $n = 2$.



$$\mathscr{B}\Delta^2 = [c, w_0, v_2] - [c, w_1, v_2] + [c, w_1, v_0] - [c, w_2, v_0] + [c, w_2, v_1] - [c, w_0, v_1].$$

$$\begin{aligned}
&\partial\mathscr{B}\Delta^2 \\
=&([c, w_0] - [c, v_2] + [w_0, v_2]) - ([c, w_1] - [c, v_2] + [w_1, v_2]) + ([c, w_1] - [c, v_0] + [w_1, v_0]) \\
&- ([c, w_2] - [c, v_0] + [w_2, v_0]) + ([c, w_2] - [c, v_1] + [w_2, v_1]) - ([c, w_0] - [c, v_1] + [w_0, v_1]) \\
=&([w_0, v_2] - [w_0, v_1]) - ([w_1, v_2] - [w_1, v_0]) + ([w_2, v_1] - [w_2, v_0]) \\
=&\mathscr{B}(\partial\Delta^2).
\end{aligned}$$

Lemma

There exists $T_{n+1} \in S_{n+1}(\Delta^n)$ for all $n \geq 0$ such that

$$\mathscr{B}_n - 1_{\Delta^n} = T_{n+1} \circ \partial\Delta^{n+1} + \partial\Delta^n \circ T_n.$$

Here $1_{\Delta^n} : \Delta^n \to \Delta^n$ is the identity, viewed as a n-chain in $S_n(\Delta^n)$.

Proof

We construct T_n inductively. $T_1 = 0$. Suppose we have found T_n. We need to find T_{n+1} such that

$$\partial(T_{n+1}) = \mathscr{B}_n - 1_{\Delta^n} - \partial\Delta^n \circ T_n.$$

Using the previous Lemma, we have

$$\begin{aligned}\partial\left(\mathscr{B}_n - 1_{\Delta^n} - \partial\Delta^n \circ T_n\right) =& \left(\mathscr{B}_n - 1_{\Delta^n} - \partial\Delta^n \circ T_n\right) \circ \partial\Delta^n \\ =& \partial\Delta^n \circ \left(\mathscr{B}_{n-1} - 1_{\Delta^{n-1}} - T_n \circ \partial\Delta^n\right) \\ =& \partial\Delta^n \circ \partial\Delta^{n-1} \circ T_{n-1} = 0.\end{aligned}$$

Therefore $\mathscr{B}_n - 1_{\Delta^n} - \partial\Delta^n \circ T_n$ is a n-cycle in $S_n(\Delta^n)$. However

$$\mathrm{H}_n(\Delta^n) = 0, \quad \forall n \geq 1$$

since Δ^n is contractible. So $\mathscr{B}_n - 1_{\Delta^n} - \partial\Delta^n \circ T_n$ is a n-boundary and T_{n+1} can be constructed. $\square$

Definition

We define the barycentric subdivision on singular chain complex by

$$\mathscr{B}^* : S_\bullet(X) \to S_\bullet(X)$$

where $\mathscr{B}^* = \mathscr{B}_n^*$ on $S_n(X)$.

Theorem

The barycentric subdivision map $\mathscr{B}^* : S_\bullet(X) \to S_\bullet(X)$ is a chain map. Moreover, it is chain homotopic to the identity map, hence a quasi-isomorphism.

Excision

Theorem (Excision)

Let $U \subset A \subset X$ be subspaces such that $\bar{U} \subset A^\circ$ (the interior of A). Then the inclusion $i : (X - U, A - U) \hookrightarrow (X, A)$ induces isomorphisms

$$i_* : \mathrm{H}_n(X - U, A - U) \simeq \mathrm{H}_n(X, A), \quad \forall n.$$

Proof

Let us call $\sigma : \Delta^n \to X$ small if

$$\sigma(\Delta^n) \subset A \quad \text{or} \quad \sigma(\Delta^n) \subset X - U.$$

Let $S'_\bullet(X) \subset S_\bullet(X)$ denote the subcomplex generated by small simplexes. The condition $\bar{U} \subset A^\circ$ implies that for any simplex $\sigma : \Delta^n \to X$, there exists a big enough k such that

$$(\mathscr{B}^*)^k(\sigma) \in S'(X).$$

Let $S'_\bullet(X,A)$ be defined by the exact sequence

$$0 \to S_\bullet(A) \to S'(X) \to S'(X,A) \to 0.$$

It is easy to see that

$$S'_\bullet(X,A) \simeq S_\bullet(X-U, A-U).$$

There is a natural commutative diagram of chain maps

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet(A) \arrow[r] \arrow[d] & S'_\bullet(X) \arrow[r] \arrow[d] & S'_\bullet(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_\bullet(A) \arrow[r] & S_\bullet(X) \arrow[r] & S_\bullet(X,A) \arrow[r] & 0
\end{tikzcd}$$

By the Five Lemma, it is enough to show

$$S'_\bullet(X) \to S_\bullet(X)$$

is a quasi-isomorphism.

- injectivity of $\mathrm{H}(S'_\bullet(X)) \to \mathrm{H}(S_\bullet(X))$.

Let α be a cycle in $S'_\bullet(X)$ and $\alpha = \partial\beta$ for $\beta \in S_\bullet(X)$. Take k big enough that $(\mathscr{B}^*)^k(\beta) \in S'(X)$. Then

$$(\mathscr{B}^*)^k(\alpha) = \partial(\mathscr{B}^*)^k(\beta).$$

Hence the class of $(\mathscr{B}^*)^k(\alpha)$ in $\mathrm{H}(S'_\bullet(X))$ is zero, so is α which is homologous to $(\mathscr{B}^*)^k(\alpha)$.

- surjectivity of $\mathrm{H}(S'_\bullet(X)) \to \mathrm{H}(S_\bullet(X))$.

Let α be a cycle in $S_\bullet(X)$. Take k big enough that $(\mathscr{B}^*)^k(\alpha) \in S'_\bullet(X)$. Then $(\mathscr{B}^*)^k(\alpha)$ is a small cycle which is homologous to α.

$\square$

Theorem

Let X_1, X_2 be subspaces of X and $X = X_1^\circ \cup X_2^\circ$. Then

$$\mathrm{H}_\bullet(X_1, X_1 \cap X_2) \to \mathrm{H}_\bullet(X, X_2)$$

is an isomorphism for all n.

Proof.

Apply Excision to $U = X - X_1, A = X_2$.

$\square$

Theorem (Mayer-Vietoris)

Let X_1, X_2 be subspaces of X and $X = X_1^\circ \cup X_2^\circ$. Then there is an exact sequence

$$\cdots \to \mathrm{H}_n(X_1 \cap X_2) \xrightarrow{(i_{1*}, i_{2*})} \mathrm{H}_n(X_1) \oplus \mathrm{H}_n(X_2) \xrightarrow{j_{1*} - j_{2*}} \mathrm{H}_n(X) \xrightarrow{\delta} \mathrm{H}_{n-1}(X_1 \cap X_2) \to \cdots$$

It is also true for the reduced homology.

Proof

Let $S_\bullet(X_1) + S_\bullet(X_2) \subset S_\bullet(X)$ be the subspace spanned by $S_\bullet(X_1)$ and $S_\bullet(X_2)$. We have a short exact sequence

$$0 \to S_\bullet(X_1 \cap X_2) \stackrel{(i_1, i_2)}{\to} S_\bullet(X_1) \oplus S_\bullet(X_2) \stackrel{j_1 - j_2}{\to} S_\bullet(X_1) + S_\bullet(X_2) \to 0.$$

Similar to the proof of Excision via barycentric subdivision, the embedding $S_\bullet(X_1) + S_\bullet(X_2) \subset S_\bullet(X)$ is a quasi-isomorphism. Mayer-Vietoris sequence follows. $\square$

Theorem

Let $A \subset X$ be a closed subspace. Assume A is a strong deformation retract of a neighborhood in X. Then the map $(X, A) \to (X/A, A/A)$ induces an isomorphism

$$\mathrm{H}_\bullet(X, A) = \tilde{\mathrm{H}}_\bullet(X/A).$$

Proof

Let U be an open neighborhood of A that deformation retracts to A. Then $\mathrm{H}_\bullet(A) \simeq \mathrm{H}_\bullet(U)$. By Five Lemma

$$\mathrm{H}_\bullet(X, A) \simeq \mathrm{H}_\bullet(X, U)$$

Since A is closed and U is open, we can apply Excision to find

$$\mathrm{H}_\bullet(X, A) \simeq \mathrm{H}_\bullet(X, U) \simeq \mathrm{H}_\bullet(X - A, U - A).$$

The same consideration applied to $(X/A, A/A)$ and U/A gives

$$\mathrm{H}_\bullet(X/A, A/A) \simeq \mathrm{H}_\bullet(X/A - A/A, U/A - A/A) = \mathrm{H}_\bullet(X - A, U - A).$$

$\square$

This Theorem in particular applies to cofibrations.

Corollary

Let $A \subset X$ be a closed cofibration. Then $\mathrm{H}_\bullet(X, A) = \tilde{\mathrm{H}}_\bullet(X/A)$.

Suspension

Let (X, x_0) be a well-pointed space. Recall that its reduced cone $C_\star X$ and reduced suspension ΣX are

$$C_\star X = \frac{X \times I}{(X \times \{0\} \cup x_0 \times I)}, \quad \Sigma X = \frac{X \times I}{(X \times \{0\} \cup X \times \{1\} \cup x_0 \times I)}.$$

Since (X, x_0) is a well-pointed, we have homotopy equivalences

$$C_\star X \simeq \frac{X \times I}{X \times \{0\}}, \quad \Sigma X \simeq \frac{X \times I}{(X \times \{0\} \cup X \times \{1\})}.$$

Theorem

Let (X, x_0) be a well-pointed space. Then $\tilde{\mathrm{H}}_n(\Sigma X) = \tilde{\mathrm{H}}_{n-1}(X)$.

Proof.

Let

$$Z = \frac{X \times I}{X \times \{0\}}, \quad Y = \frac{X \times I}{(X \times \{0\} \cup X \times \{1\})} = Z/X.$$

Since Z is contractible, the homology exact sequence associated to the pair $X \subset Z$ implies

$$\tilde{\mathrm{H}}_n(Z, X) = \tilde{\mathrm{H}}_{n-1}(X).$$

It follows that

$$\tilde{\mathrm{H}}_n(\Sigma X) = \tilde{\mathrm{H}}_n(Y) = \tilde{\mathrm{H}}_n(Z/X) = \tilde{\mathrm{H}}_n(Z, X) = \tilde{\mathrm{H}}_{n-1}(X).$$

□

Proposition

The reduced homology of the sphere S^n is given by

$$\tilde{\mathrm{H}}_k(S^n) = \begin{cases} \mathbb{Z} & k = n \\ 0 & k \neq n \end{cases}$$

In particular, spheres of different dimensions are not homotopy equivalent.

Proof.

This follows from the previous theorem and $S^n = \Sigma^n S^0$ where $S^0 = \{\pm 1\}$ consists of two points. $\square$

Applications

Proposition

If $m \neq n$, then $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

Proof.

Assume $f: \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism. Then f induces a homeomorphism

$$\mathbb{R}^m - \{p\} \to \mathbb{R}^n - \{f(p)\}$$

hence a homotopy equivalence between S^{m-1} and S^{n-1}. Contradiction. $\square$

Definition

A continous map $f\colon S^n \to S^n$ $(n \geq 0)$ has degree d, denoted by $\deg(f) = d$, if

$$f_* : \tilde{\mathrm{H}}_n(S^n) = \mathbb{Z} \to \tilde{\mathrm{H}}_n(S^n) = \mathbb{Z}$$

is multiplication by d.

Lemma

Let $f, g : S^n \to S^n$ be continuous maps.

1. $\deg(f \circ g) = \deg(f) \deg(g)$.
2. If $f \simeq g$ are homotopic, then $\deg(f) = \deg(g)$
3. If f is a homotopy equivalence, then $\deg(f) = \pm 1$.

Proof.

All the statements follow from the fact that H_n defines a functor

$$\mathrm{H}_n : \mathbf{hTop} \to \mathbf{Group}.$$

$\square$

Proposition

Let $r : S^n \to S^n, (x_0, \cdots, x_n) \to (-x_0, x_1, \cdots, x_n)$ be the reflection. Then $\deg(r) = -1$.

Proof.

We prove by induction on n. This is true for $n = 0$. Assume the case for $n - 1$. We find an isomorphism

$$\tilde{\mathrm{H}}_n(S^n) = \tilde{\mathrm{H}}_n(D^n/S^{n-1}) = \tilde{\mathrm{H}}_n(D^n, S^{n-1}) \xrightarrow{\delta} \tilde{\mathrm{H}}_{n-1}(S^{n-1}).$$

This leads to the commutative diagram

$$\begin{tikzcd}
\tilde{\mathrm{H}}_n(S^n) \arrow[r, "\delta"] \arrow[d, "r_*"] & \tilde{\mathrm{H}}_{n-1}(S^{n-1}) \arrow[d, "r_*"] \\
\tilde{\mathrm{H}}_n(S^n) \arrow[r, "\delta"] & \tilde{\mathrm{H}}_{n-1}(S^{n-1})
\end{tikzcd}$$

This proves the case for n.

Theorem (Hairy Ball Theorem)

S^n has a nowhere vanishing tangent vector field $\Leftrightarrow$ n is odd.

Proof.

If n is odd, we construct

$$v(x_0, \cdots, x_n) = (-x_1, x_0, -x_3, x_2, \cdots).$$

Conversely, assume v is no-where vanishing vector field. Let

$$f: S^n \to S^n, \quad x \to \frac{v(x)}{|v(x)|}.$$

The map

$$F: S^n \times I \to S^n, \quad F(x,t) = \cos(\pi t)x + \sin(\pi t)f(x)$$

defines a homotopy between the identity map 1 and the antipodal map σ. It follows that $\deg(\sigma) = 1 \implies n = \text{odd}$.

Theorem (Brower's Fixed Point Theorem)

Any continuous map $f: D^n \to D^n$ has a fixed point.

Proof.

Assume f has no fixed point. Define

$$r: D^n \to S^{n-1}$$

where $r(p)$ is the intersection of ∂D^n with the ray starting from $f(p)$ pointing toward p. Then r defines a retract of $S^{n-1} \hookrightarrow D^n$. This implies $\mathrm{H}_\bullet(D^n) = \mathrm{H}_\bullet(S^{n-1}) \oplus \mathrm{H}_\bullet(D^n, S^{n-1})$. Contradiction. $\square$